triple reference class. For these, according to the rule of elimination, there are only six independent equations of the form

$$P(A,D) = P(A.B,D) = P(A.B,C) \cdot P(A.B.C,D) + [1 - P(A.B,C)] \cdot P(A.B.\bar{C},D) \tag{14}$$

The probabilities having a triple reference class are, therefore, not determined by the probabilities having a single or a double reference class, and thus (13) does not follow from (12).

An example of a case for which (12) is valid, but not (13), is provided by the sequences

$$\begin{array}{l} A\ A\ A\ A\ A\ A\ A\ A\ \ldots \\ B\ \bar{B}\ B\ \bar{B}\ B\ \bar{B}\ B\ \bar{B}\ \ldots \\ C\ C\ \bar{C}\ \bar{C}\ C\ C\ \bar{C}\ \bar{C}\ \ldots \\ D\ \bar{D}\ \bar{D}\ D\ D\ \bar{D}\ \bar{D}\ D\ \ldots \end{array} \tag{15}$$

for which the first part written down is to be repeated periodically in the same order. Here all the probabilities (12) are equal to $\frac{1}{2}$. But $P(A.B.C,D)$ is equal to 1; so is $P(A.\bar{B}.\bar{C},D)$, and so on.

Sequences for which, apart from the relations (12), the relations (13) are fulfilled, are called *completely independent*. This notation applies similarly for a greater number of sequences.

§ 24. Complete Probability Systems

In § 16 the assumption of a compact sequence A was introduced and shown to be convenient for the frequency interpretation, because it leads to the simple formula (4, § 16). It is possible to introduce this assumption by a logical device that makes its truth analytic: by replacing the class A by the universal class $A \vee \bar{A}$. The condition $x_i \epsilon A \vee \bar{A}$ is then tautologically satisfied for every element x_i.

To simplify the notation we introduce the rule that the universal class may be omitted in the first term of a probability expression. This rule is expressed by the definition

$$P(B) =_{Df} P(A \vee \bar{A},B) \tag{1}$$

The probability $P(B)$ may be called an *absolute probability*, in contradistinction to the relative probabilities so far considered. An absolute probability can be regarded as a relative probability the reference class of which is the universal class.

If the statement $x_i \epsilon A$ is true for all x_i, though not analytic, the class A, for this sequence, is equivalent to the universal class $A \vee \bar{A}$. If a sequence is compact in A, the indication of the class A may therefore be omitted, and the probabilities may be treated as absolute probabilities.

The axioms and theorems of the calculus are transferred by the definition (1) to absolute probabilities. We find, for instance,

$$P(B) + P(\bar{B}) = 1 \tag{2}$$

$$P(B \vee C) = P(B) + P(C) - P(B.C) \tag{3}$$

$$P(B.C) = P(B) \cdot P(B,C) = P(C) \cdot P(C,B) \tag{4}$$

$$P(C) = P(B) \cdot P(B,C) + P(\bar{B}) \cdot P(\bar{B},C) \tag{5}$$

$$P(B \vee C,D) = \frac{P(B) \cdot P(B,D) + P(C) \cdot P(C,D) - P(B) \cdot P(B,C) \cdot P(B.C,D)}{P(B) + P(C) - P(B) \cdot P(B,C)} \tag{6}$$

and for exclusive events

$$P(B \vee C,D) = \frac{P(B) \cdot P(B,D) + P(C) \cdot P(C,D)}{P(B) + P(C)} \tag{7}$$

These special forms follow from the general forms (7, § 13), (8, § 20), (3, § 14), (2, § 19), (16, § 22), (3, § 22), when $A \vee \bar{A}$ is substituted for A. But it is not possible conversely to derive the general forms from the special forms; the latter hold only for the universal class as reference class, whereas the former hold for all reference classes. For the treatment of the relative probabilities occurring in the above formulas, therefore, formulas in terms of the general reference class A must be used.[1]

If two classes B and C are considered, the *complete system of probabilities* pertaining to them is given by the probabilities

$$\begin{array}{cccc} P(B) & P(\bar{B}) & P(C) & P(\bar{C}) \\ P(B,C) & P(\bar{B},C) & P(C,B) & P(\bar{C},B) \\ P(B,\bar{C}) & P(\bar{B},\bar{C}) & P(C,\bar{B}) & P(\bar{C},\bar{B}) \end{array} \tag{8}$$

The probabilities of the second and third line will be called *mutual probabilities*. The twelve values (8) are determined by the three fundamental probabilities

$$P(B) \qquad P(C) \qquad P(B,C) \tag{9}$$

which are the analogues of the three fundamental probabilities $P(A,B)$, $P(A,C)$, $P(A.B,C)$, introduced in (7′, § 21). The computation is made by

[1] If formulas containing the general reference class A are to be derivable from the corresponding formulas written in the notation by absolute probabilities, a particular rule of substitution must be introduced; see rule α, § 82.

the use of the relations (2)–(5); among these, (4) supplies the value $P(C,B)$; (5), the values with negated terms in the reference class.

It was mentioned above that the choice of these values as fundamental probabilities is a matter of convention, and that three other independent values could be used for the same purpose. From this point of view it is of some interest to select three values from the second line in (8) as fundamental probabilities. This line contains the two *affirmative* terms $P(B,C)$ and $P(C,B)$, which contain no negation signs, and the two terms of *negative reference* $P(\bar{B},C)$ and $P(\bar{C},B)$, which contain negation signs for reference classes. These four probabilities are not independent, but are connected by the relation

$$\frac{P(B,C)}{P(C,B)} \cdot \frac{P(\bar{C},B)}{P(\bar{B},C)} = \frac{1 - P(B,C) - P(\bar{C},B)}{1 - P(C,B) - P(\bar{B},C)} \tag{10}$$

This relation is derived as follows. We introduce the abbreviations

$$P(B) = b \qquad P(C) = c \qquad P(B,C) = c_1 \qquad P(\bar{B},C) = c_2$$
$$P(C,B) = b_1 \qquad P(\bar{C},B) = b_2 \tag{11}$$

Applying (4) to the three forms $P(B.C)$, $P(\bar{B}.C)$, $P(B.\bar{C})$, we construct the three relations

$$bc_1 = cb_1 \qquad c(1 - b_1) = c_2(1 - b) \qquad b(1 - c_1) = b_2(1 - c) \tag{12}$$

Solving the first two relations for b and c, we have

$$b = \frac{b_1 c_2}{c_1(1 - b_1) + b_1 c_2} \qquad c = \frac{c_1 c_2}{c_1(1 - b_1) + b_1 c_2} \tag{13}$$

Inserting these values in the third relation (12), we find

$$\frac{c_1 b_2}{b_1 c_2} = \frac{b_2 + c_1 - 1}{c_2 + b_1 - 1} \tag{14}$$

which is the relation (10).

Equations (13) determine the absolute probabilities as functions of the three mutual probabilities $P(B,C)$, $P(C,B)$, $P(\bar{B},C)$, and can be written in the form

$$P(B) = \frac{P(C,B) \cdot P(\bar{B},C)}{P(B,C)[1 - P(C,B)] + P(C,B) \cdot P(\bar{B},C)}$$
$$P(C) = \frac{P(B,C) \cdot P(\bar{B},C)}{P(B,C)[1 - P(C,B)] + F(C,B) \cdot P(\bar{B},C)} \tag{15}$$

These results show that all the probabilities (8) are determined by the three mutual probabilities $P(B,C)$, $P(C,B)$, $P(\bar{B},C)$. Exception is to be made for the case that the denominator of (15) (which represents the determinant of the corresponding set of linear equations) vanishes. This is the case, in particular, for exclusive classes B and C, that is, for $P(B,C) = P(C,B) = 0$; (15) then gives the indeterminate form $\frac{0}{0}$.

It is possible to construct a formula that is not subject to this degeneration when the probability $P(\bar{C},B)$ is included in the arguments. For $P(B,C) = P(C,B) = 0$, formula (10) supplies the form $\frac{0}{0}$, and the four values of the second line of (8) are no longer connected by a restrictive condition; thus the last term of this line can be added as an independent parameter. For the derivation we use (5), first with B and C interchanged, then in the form given, thus arriving at the two equations

$$b = cb_1 + (1 - c)b_2 \qquad c = bc_1 + (1 - b)c_2 \tag{16}$$

which, solved for b and c, give the results

$$b = \frac{c_2b_1 + b_2(1 - c_2)}{1 - (b_1 - b_2)(c_1 - c_2)} \qquad c = \frac{b_2c_1 + c_2(1 - b_2)}{1 - (b_1 - b_2)(c_1 - c_2)} \tag{17}$$

Introducing the exclusion condition $b_1 = c_1 = 0$, we find

$$b = \frac{b_2(1 - c_2)}{1 - b_2c_2} \qquad c = \frac{c_2(1 - b_2)}{1 - b_2c_2} \tag{18}$$

These formulas can be written

$$P(B) = \frac{P(\bar{C},B)[1 - P(\bar{B},C)]}{1 - P(\bar{B},C) \cdot P(\bar{C},B)} \qquad P(C) = \frac{P(\bar{B},C)[1 - P(\bar{C},B)]}{1 - P(\bar{B},C) \cdot P(\bar{C},B)} \tag{19}$$

They determine the absolute probabilities in terms of the mutual probabilities for the exclusive classes B and C.

The two affirmative mutual probabilities $P(B,C)$ and $P(C,B)$ are connected by the relation

$$\frac{P(B,C)}{P(C,B)} = \frac{P(C)}{P(B)} \tag{20}$$

which follows from (4). Corresponding relations hold for negated mutual probabilities; they follow from (20) by the substitution of $\bar{B}$ for B and so on.

In many applications the two absolute probabilities $P(B)$ and $P(C)$ are unknown, and only the two affirmative mutual probabilities $P(B,C)$ and $P(C,B)$ are given. These values are subject to no restrictions other than that their values be between 0 and 1. The two probabilities are sometimes combined in a *mutual probability implication*, which is written in the implicative notation

$$(B \underset{q\;\;p}{\rightleftharpoons} C) \tag{21}$$

This is equivalent to the conjunction

$$(B \underset{p}{\ni} C) \,.\, (C \underset{q}{\ni} B) \tag{22}$$

As explained, the two values p and q are not subject to any connecting condition. It is even possible that $p = 1$ and $q = 0$ without B or C being empty. The corresponding general implications $(B \supset C)$ and $(C \supset \bar{B})$ are compatible only if B is empty, since we can derive from them, by the transitivity of the implication, the relation $(B \supset \bar{B})$. There is no such consequence for the probability implications because the probability 1 is not equivalent to certainty. Thus $P(C,B) = 0$ does not exclude the possibility that C is sometimes accompanied by B.

If the two mutual probabilities are given, the values of the absolute probabilities and those of the probabilities of negative reference are not determined. Only the ratio of the absolute probabilities is determined, according to (20). But it is possible to compute some other probabilities. First, we can replace the relation (20), which includes absolute probabilities, by a corresponding relation for relative probabilities. For this derivation, the relations (2)–(7) are not sufficient, and we must return to the notation in terms of the general reference class A. We apply (4, § 21), substitute for A the disjunction $B \vee C$, and use the tautological equivalences

$$([B \vee C].B \equiv B) \qquad ([B \vee C].C \equiv C) \tag{23}$$

We thus arrive at the formula

$$\frac{P(B,C)}{P(C,B)} = \frac{P(B \vee C,C)}{P(B \vee C,B)} \tag{24}$$

The expressions $P(B \vee C,B)$ and $P(B \vee C,C)$ may be called *disjunctive weights;* they determine the weight with which either of the terms B or C occurs in their mutual disjunction. Formula (24) states that the ratio of the mutual probabilities is equal to the ratio of the disjunctive weights.

There is a further relation, which makes it possible, in combination with (24), to determine the disjunctive weights in terms of the affirmative mutual probabilities. We have, with the general rule of addition in the A-notation,

$$P(B \vee C,B \vee C) = 1 = P(B \vee C,B) + P(B \vee C,C) - P(B \vee C,B.C) \tag{25}$$

The last term is transformed with (23) into

$$\begin{aligned} P(B \vee C,B.C) &= P(B \vee C,B) \cdot P([B \vee C].B,C) \\ &= P(B \vee C,B) \cdot P(B,C) \end{aligned} \tag{26}$$

Introducing this result in (25) and substituting for $P(B \vee C,C)$ the value resulting from (24), we arrive at an equation, which, solved for $P(B \vee C,B)$, gives the result

$$P(B \vee C,B) = \frac{P(C,B)}{P(B,C) + P(C,B) - P(B,C) \cdot P(C,B)} \tag{27}$$

This relation will be called the *general rule of the disjunctive weight.* It determines the disjunctive weight as a function of the affirmative mutual probabilities.

It is easily seen that the disjunctive weight of C is given by a similar expression, resulting from (27) when $P(B,C)$ is put for $P(C,B)$ in the numerator. The probability of the product results from (26) in the form

$$P(B \vee C,B.C) = \frac{P(B,C) \cdot P(C,B)}{P(B,C) + P(C,B) - P(B,C) \cdot P(C,B)} \tag{28}$$

As for (15), a qualification must be added. Formulas (27)–(28) depend on the condition that at least one of the two mutual probabilities is > 0. It follows that for exclusive classes the disjunctive weights are not determined by the affirmative mutual probabilities.

As before, a computation for exclusive classes is made possible by the use of mutual probabilities of negative reference. From (7) we derive, substituting B for D and putting $P(C,B) = 0$ because of the exclusion condition,

$$P(B \vee C,B) = \frac{P(B)}{P(B) + P(C)} \tag{29}$$

With the application of (19) we find

$$P(B \vee C,B) = \frac{P(\bar{C},B)[1 - P(\bar{B},C)]}{P(\bar{B},C) + P(\bar{C},B) - 2P(\bar{B},C) \cdot P(\bar{C},B)} \tag{30}$$

This formula, which holds only for exclusive disjunctions, will be called the *special rule of the disjunctive weight.* Since the mutual probabilities of negative reference used in the formula are sufficient to determine the absolute probabilities, a knowledge of the disjunctive weights, for exclusive disjunctions, is inseparable from a knowledge of the absolute probabilities.

We turn now to probability relations between three classes B_1,B_2,B_3. The complete probability system, written only for affirmative terms, is given here by the probabilities

$$\begin{array}{ccc} P(B_1) & P(B_2) & P(B_3) \\ P(B_1,B_2) & P(B_2,B_3) & P(B_3,B_1) \\ P(B_2,B_1) & P(B_3,B_2) & P(B_1,B_3) \\ P(B_1.B_2,B_3) & P(B_2.B_3,B_1) & P(B_3.B_1,B_2) \end{array} \tag{31}$$

The other forms result by substitution of $\bar{B}_1$ and so on, in these expressions. The probabilities of the last line will be called *compound mutual probabilities*. Those of the second and third lines are then called *simple mutual probabilities*. Note that the values (31) are not subject to restrictive conditions: it is not required that the three classes be exclusive or independent or that they form a complete disjunction.

For any two simple mutual probabilities, formula (4) leads to the relation

$$\frac{P(B_i,B_k)}{P(B_k,B_i)} = \frac{P(B_k)}{P(B_i)} \tag{32}$$

The probabilities of the third line of (31) are thus determined by those of the first and second lines. Probabilities having $\bar{B}_i$ or $\bar{B}_k$ in the reference class are derivable from the affirmative terms by means of (5).

The three compound mutual probabilities are connected by the relations, following from the rule of the product,

$$\frac{P(B_i . B_k,B_m)}{P(B_i . B_m,B_k)} = \frac{P(B_i,B_m)}{P(B_i,B_k)} \tag{33}$$

The three relations resulting for $m = 1$, $k = 2$; $m = 2$, $k = 3$; and $m = 3$, $k = 1$, are not independent, because the last is easily seen to be a consequence of the other two. Thus (33) represents two independent relations. If one of the affirmative compound mutual probabilities is given, the other two are thus determined. Probabilities with terms $\bar{B}_i$ in the reference class are computed from the affirmative terms by means of the relations (5) and (10, § 19).

The complete probability system for three classes is thus determined by the six values of the first two lines of (31) and, besides, one of the compound values of the last line of (31), that is, by seven independent probability values.

If the absolute probabilities $P(B_1)$, $P(B_2)$, $P(B_3)$ are unknown, the six values of the simple mutual probabilities in the second and third lines of (31) cannot be assumed arbitrarily, but are connected by the relation

$$\frac{P(B_1,B_2)}{P(B_2,B_1)} \cdot \frac{P(B_2,B_3)}{P(B_3,B_2)} \cdot \frac{P(B_3,B_1)}{P(B_1,B_3)} = 1 \tag{34}$$

which follows by the use of (32) and can be written in the form[2]

$$P(B_1,B_2) \cdot P(B_2,B_3) \cdot P(B_3,B_1) = P(B_1,B_3) \cdot P(B_3,B_2) \cdot P(B_2,B_1) \tag{35}$$

Only five of these values, therefore, are independent. Formula (35) will be called the *rule of the triangle*. It states that in a triangle $B_1B_2B_3$ the product of the three mutual probabilities is the same, whether we go clockwise or

[2] This relation was pointed out by Norman Dalkey, "The Plurality of Language Structures," doctoral dissertation, University of California at Los Angeles, 1942.

counterclockwise around the triangle. For two events there is no such dependence of mutual probabilities, because in this case there is only one direction for the "round trip". The distinction of two such directions begins with three events.

The relation (35) has a simple explanation in the frequency interpretation; it represents the identity

$$\frac{N^n(B_1.B_2)}{N^n(B_1)}\cdot\frac{N^n(B_2.B_3)}{N^n(B_2)}\cdot\frac{N^n(B_3.B_1)}{N^n(B_3)} = \frac{N^n(B_1.B_3)}{N^n(B_1)}\cdot\frac{N^n(B_3.B_2)}{N^n(B_3)}\cdot\frac{N^n(B_2.B_1)}{N^n(B_2)} \tag{36}$$

The rule of the triangle (35) is automatically satisfied if the absolute probabilities in combination with the second line of (31) are used for the determination of the values of the third line. But if the absolute probabilities are not used and, instead, five of the simple mutual probabilities are assumed arbitrarily, they are subject to the numerical restriction

$$P(B_1,B_2)\cdot P(B_2,B_3)\cdot P(B_3,B_1) \leqq P(B_3,B_2)\cdot P(B_2,B_1) \tag{37}$$

which formulates the condition $P(B_1,B_3) \leqq 1$ for computation of this probability from (35). This inequality is to be added to the inequalities (15, § 19).

The following special conditions can be derived. If $P(B_1,B_2) = 1$ and $P(B_2,B_3) = 1$, it follows that $P(B_1,B_3) = 1$ if $P(B_2,B_1) > 0$. This transitivity is shown by the considerations added to (7, § 19). Another rule of transitivity is as follows: if transitivity holds in one direction of the triangle, it also holds in the other. This theorem, which applies, too, when the probabilities are < 1, is derivable from (35), because when we put there

$$P(B_1,B_3) = P(B_1,B_2)\cdot P(B_2,B_3)$$

we have

$$P(B_3,B_1) = P(B_3,B_2)\cdot P(B_2,B_1)$$

The values (37) determine the ratios of the absolute probabilities, according to (32). If a further condition is added, for instance, that the disjunction of the three classes be complete, the absolute probabilities are determinable. Instead of such a condition for the absolute probabilities, it is sufficient to give one simple mutual probability of negative reference. The computation of the absolute probabilities then follows the methods developed for two classes.

If, besides the values (37), one compound probability and one simple mutual probability of negative reference are given, all the other probabilities can be computed by the methods developed for two classes.

For three classes, the problem of a disjunctive reference class offers particular interest. The problem will first be treated for nonexclusive reference classes B_1 and B_2, for which case it can be solved in terms of affirmative mutual probabilities. When we insert the values (15) in (6), the denominator of (15) drops out and the term $P(\bar{B},C)$, which occurs in every term, can be canceled. Putting B_1, B_2, B_3, respectively, for B, C, D, we arrive at the formula

$$P(B_1 \vee B_2,B_3) = \frac{P(B_1,B_2) \cdot P(B_2,B_3) + P(B_2,B_1) \cdot P(B_1,B_3) - P(B_1,B_2) \cdot P(B_2,B_1) \cdot P(B_1 . B_2,B_3)}{P(B_1,B_2) + P(B_2,B_1) - P(B_1,B_2) \cdot P(B_2,B_1)} \tag{38}$$

The formula differs from the general rule of reduction, in the forms (16, § 22) or (6), in that it includes neither absolute probabilities nor a term A common to all reference classes. Instead, the terms B_1 and B_2 of the disjunction are distributed into the first terms of the expressions on the right; we therefore call (38) the *general rule of distributive reference.* The occurrence of the term $P(B_1 . B_2,B_3)$ shows that the solution requires one compound mutual probability; but all the terms are affirmative mutual probabilities.

For exclusive classes, again, a different solution is necessary, because, for $P(B_1,B_2) = P(B_2,B_1) = 0$, formula (38) gives the form $\frac{0}{0}$. As before, the problem is solved by the use of probabilities of negative reference. Starting with (7), we insert the values of $P(B)$ and $P(C)$ from (19); we then substitute B_1, B_2, B_3, respectively, for B, C, D, and arrive at the result

$$P(B_1 \vee B_2,B_3) = \frac{P(B_1,B_3) \cdot P(\bar{B}_2,B_1) \cdot [1 - P(\bar{B}_1,B_2)] + P(B_2,B_3) \cdot P(\bar{B}_1,B_2) \cdot [1 - P(\bar{B}_2,B_1)]}{P(\bar{B}_1,B_2) + P(\bar{B}_2,B_1) - 2P(\bar{B}_1,B_2) \cdot P(\bar{B}_2,B_1)} \tag{39}$$

This is the *special rule of distributed reference,* which holds only for exclusive disjunctions. It does not require the use of compound mutual probabilities, but presupposes terms of negative reference. Note that, in contradistinction to previous theorems to which similar names were assigned (§§ 14, 20, 22), the two special rules (30) and (39) do not follow from the two general rules (27) and (38) as special cases, but require separate derivations.

The considerations can be extended to n classes $B_1 \ldots B_n$. For every subset of m classes there exist compound mutual probabilities of the form $P(B_{k_1} \ldots B_{k_{m-1}},B_{k_m})$. They are connected with those of the next lower subset by the relations

$$\frac{P(B_{k_1} \ldots B_{k_{m-1}},B_{k_m})}{P(B_{k_1} \ldots B_{k_{m-2}} . B_{k_m},B_{k_{m-1}})} = \frac{P(B_{k_1} \ldots B_{k_{m-2}},B_{k_m})}{P(B_{k_1} \ldots B_{k_{m-2}},B_{k_{m-1}})} \tag{40}$$

which express the rule of the product. Given one probability of the subset, all the others are thus determined in terms of those of the next lower subset.

Probabilities having a term $\bar{B}_{k_r}$ in the reference class are determined by the use of the rule of elimination (10, § 19).

The total number μ of independent probabilities determining the complete probability system for n classes is computed as follows. First, the n absolute probabilities $P(B_i)$ can be given independently. Second, of each subset of m classes one probability must be given. This includes the case $m = 2$, for which we have the simple mutual probabilities; for every combination $P(B_i,B_k)$, the converse probability follows from (32) in terms of the absolute probabilities. The relation (32) is not restricted to three classes and is a special case of (40) resulting for $m = 2$. The number of subsets of m classes among n classes being $\binom{n}{m}$ we find, since $n = \binom{n}{1}$,

$$\mu = \sum_{m=1}^{n} \binom{n}{m} = 2^n - 1 \tag{41}$$

using the familiar theorem for binomial coefficients

$$\sum_{m=0}^{n} \binom{n}{m} = 2^n \tag{42}$$

For $n = 2$ we have $\mu = 3$, according to (9). For $n = 3$ we have $\mu = 7$, in correspondence with the above result.

The number ν of affirmative probabilities can be found as follows. For every subset of m classes, there are m affirmative probabilities, which result when, one after another, each of the classes is chosen as attribute class. This is true, too, for $m=2$ and $m=1$. There being $\binom{n}{m}$ subsets of m terms, we have

$$\nu = \sum_{m=1}^{n} \binom{n}{m} m = \sum_{m=1}^{n} \frac{n!}{m!\,(n-m)!} \cdot m$$

$$= n \sum_{m=1}^{n} \frac{(n-1)!}{(m-1)!(n-m)!} = n \sum_{m=1}^{n} \frac{(n-1)!}{(m-1)![(n-1)-(m-1)]!}$$

$$= n \sum_{m=1}^{n} \binom{n-1}{m-1} = n \sum_{m-1=0}^{n-1} \binom{n-1}{m-1} = n \cdot 2^{n-1} \tag{43}$$

For $n = 1$ we have $\nu = 1$; for $n = 2$, $\nu = 4$; for $n = 3$, $\nu = 12$, in correspondence with (9) and (31).

The number ρ of probabilities of the complete system, all being of the form occurring in (40), but including both affirmative and negated terms, is computed as follows. Each affirmative probability of m terms contributes 2^m into the total, since each term can be written once without, and once with, a negation sign. So we have, using the preceding transformation of the sum,

$$\rho = \sum_{m=1}^{n} \binom{n}{m} m \cdot 2^m = n \sum_{m-1=0}^{n-1} \binom{n-1}{m-1} \cdot 2^m = 2n \sum_{m-1=0}^{n-1} \binom{n-1}{m-1} \cdot 2^{m-1}$$

$$= 2n(2+1)^{n-1} = 2n \cdot 3^{n-1} \tag{44}$$

For the transition to the second line we use the binomial theorem

$$(p+q)^n = \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} \tag{45}$$

choosing $p = 2$, $q = 1$, and putting $n - 1$ for n and $m - 1$ for m. For $n = 1$ we have $\rho = 2$; for $n = 2$, $\rho = 12$ (see 8); for $n = 3$, $\rho = 54$.

There are many applications for the relations developed for three classes. Let B_1 be a symptom of illness; B_2, a certain disease; B_3, the case of death. The simple mutual probabilities may be known from statistics; the relation (35) shows that only five are to be ascertained, the sixth being determinable. Furthermore, one of the compound probabilities must be ascertained, for instance, $P(B_1 . B_2, B_3)$. When these values are known, all statistical questions referring to the three classes are answerable except those referring to absolute probabilities or probabilities of negative reference. A psychological application obtains when B_1 means a certain stimulus; B_2, a perception; B_3, a certain reaction of a person.

§ 25. Remarks Concerning the Mathematical Formalization of the Probability Calculus

Having carried through, to a large extent, the formalization of the calculus of probability, we are now free to discuss this procedure from a logical viewpoint. The "logification" by which this construction of the calculus was introduced has, in the meantime, been transformed into a "mathematization", a notation in which the logical operations are restricted to the inner part of the P-symbols. The resulting complexes of the P-symbols, into which these symbols enter as units, have the character of mathematical equations. Thus the probability calculus acquires a form that is convenient for the purpose of carrying out calculations.

This manner of writing—the mathematical notation—has the disadvantage that it cannot express certain relations of a nonmathematical kind that hold within the probability calculus. There are three different forms of such relations:

1. The dependence of a mathematical equation on the validity of another mathematical equation, that is, the implication between equations. An exam-