BELIEF, EXISTENCE, and MEANING

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a defined term or phrase. Here also various patterns of belief may be introduced, some of them depending upon the user's familiarity with the definitions of *L*.

Some readers might object that we have tarried too long over the forms for handling '*X* believes so and so'. The forms are crucial, pivotal forms in the logic of belief, however, and additional ones will be presented as we go on. The choice of a form here is determined to a large extent by one's philosophy of logic, and in turn determines to a large extent a whole philosophic view. Failure to be clear about the forms here, and to choose a satisfactory one, increases the likelihood of failure or confusion in other areas of philosophy.

CHAPTER VI

The Philosophic Import of Virtual Classes

There has been occasional reference in the above to virtual classes, and there will be further reference below, where fundamental use of them will be made. What are they, why are they philosophically interesting, and in particular, what role do they play in the logic of belief? In the present chapter the attempt is made to answer these questions to some extent.

Our language is not designed to speak of virtual classes, for there are no such entities. A virtual class in fact is a mere fiction; like the ghost of Hamlet's father, 'tis here, 'tis here, 'tis gone—mostly gone. Curiously, however, discourse concerning such entities, although a mere *manière de parler*, can be made as exact as one wishes and is subject to precise logical laws. Even more curiously, such discourse is not only useful but also conceptually of such interest that it may be viewed as constituting the very nerve of first-order logic. In fact we can almost go so far as to say that first-order logic *is* the theory of virtual classes and relations in a kind of notational disguise.

Virtual classes are not values for variables in any way, shape, or form, whereas real classes are. Here in sum is the difference. However, an enormous difference it is and one perhaps not suffiicently recognized either by logicians or analytic philosophers. Real classes are values for variables in some suitable class-theoretic formalism—and likewise for sets, if one wishes to distinguish in some way between sets and classes. Virtual classes, however, are never values for variables in any formalism whatsoever. Nor could any formalism be constructed in such a way that they could be. Virtual classes are classes *als ob,* and any attempt to make them real fundamentally alters their character.

The subject has been brought to the fore again in the introductory sections of Quine's *Set Theory and Its Logic.* There, attention centers upon virtual-class and virtual-relation theory only insofar as they are of interest for the subsequent "real thing." Later, Quine merges the two in a most skillful way. Virtual theory, however, is not merely a tool for mathematics, and the main motives for interest in it are surely philosophical. Quine notes only a few, but there are many more.

First let us sketch briefly the logic of virtual classes and relations in Section 1, and then reflect in Section 2 upon why it is thought to be of fundamental philosophic interest. In Section 3 a list of *useful notions* definable in terms of virtual classes is given. In Section 4 we reflect upon *virtual classes of virtual classes,* and the like. Some comments concerning real sets and classes are given in Section 5. Finally, in Section 6 we return to the logic of belief and, in terms now of virtual classes, an improved definition of 'B' is given.

Let us not worry about who first thought up virtual classes or, more precisely, formulated an exact notation for them with appropriate laws. The theory of virtual classes and relations was so called in Quine's lectures in Brazil in 1942.[1] It played a major role in the author's thesis of 1941 (published in 1943), written under the guidance of Frederic Fitch.[2] A full development of the theory was given in the author's Bonn lectures in 1960, similar to that suggested on pages 15–27 of *Set Theory and Its Logic.*

1. Notation. The theory of virtual classes and relations is primarily a matter of notation, so it is to notational matters that we should turn for a moment.

Let '(—*x*—)', as above, be some formula of *L* containing '*x*' as its only free variable. The formulate of *L* are either atomic or (recursively) built up out of atomic formulae by means of 'V', '~', '(*x*)', '(E*x*)' and so on. Now the formula '(—*x*—)' may be either atomic or of any complexity. No matter which, it in effect predicates of the individual *x* such and such a property, or (equivalently) says that *x* is one among the individuals having that property. Let us collect *all* such individuals and form an expression for this collection. We can then use 'ϵ', the usual symbol for 'is a member of,' together with this expression, to say

[1] *O Sentido da nova lógica* (São Paulo: Martins, 1944), Section 51.

[2] "A Homogeneous System for Formal Logic," *The Journal of Symbolic Logic* 8 (1943), 1–23.

that *x* is one of this collection. In other words we can define the whole context

(1) '*x* ϵ *y*∍(—*y*—)'

or, equivalently, essentially in Quine's notation,

(2) '*x* ϵ {*y*:—*y*—}'.

(1) or (2) are defined as wholes, so that the 'ϵ' here and the expressions '*y*∍(—*y*—)' or '{*y*:—*y*—}' have no meaning in isolation. With (1) or (2) as a definiendum, the required definiens is merely

'(—*x*—)',

where '(—*x*—)' differs from '(—*y*—)' appropriately.

Any expression of the form '*y*∍(—*y*—)' or '{*y*:—*y*—}' is called 'a one-place *abstract*'. The use of the inverted epsilon to form abstracts is akin to that of Peano. The colon here, on the other hand, is often preferred by mathematicians, and there are still other variant notations in common use.

One variant, specially to be recommended perhaps, may be given in which no use is made of 'ϵ' and the abstract occurs to the left of the argument variable. Thus,

(3) '*y*∍(—*y*—) *x*'

or

(4) '{*y*:—*y*—}*x*'

may be used in place of (1) or (2). These notations have the advantage of exhibiting abstracts in the place of primitive predicates. Thus, where 'P' is a primitive one-place predicate of *L*, 'P*x*' and (3) or (4) have essentially the same form. (3) and (4) thus enable us to see clearly that formulae containing complex predicates may be given in effect the same form as atomic formulae containing only primitive predicates.

The notational legerdemain seemingly achieved by (1) or (2) (or by (3) or (4)) enables us to speak as though we were speaking of classes. Of course we are not. Everything ostensibly said in terms of classes is in fact said in terms merely of the primitive notation of *L*, which has only individuals as values for its variables.

The foregoing definition yields immediately the three logical principles:

(5) $$\text{'}x \in y \ni (\text{—}y\text{—}) \supset (\text{—}x\text{—})\text{'},$$

(6) $$\text{'}(\text{—}x\text{—}) \supset x \in y \ni (\text{—}y\text{—})\text{'},$$

and

(7) $$\text{'}x \in y \ni (\text{—}y\text{—}) \equiv (\text{—}x\text{—})\text{'}.$$

Quine calls (7) 'the *principle of concretion*', but it has frequently been called 'the *principle of abstraction*'. Actually (5) should perhaps more aptly be called 'the principle of concretion' because it enables us to pass, as it were, from an expression containing an abstract to one that does not. (6), on the other hand, enables us to pass from a concrete form, as it were, to a form containing an abstract, and hence is aptly a principle of abstraction.

Note that (1) and (2) are allowed to contain no free variable other than '*x*'. Suppose '(—*y*—)' were allowed to contain some free occurrence of a variable '*z*' other than '*y*'. The resulting abstract '$y \ni$(—*y*—)', would no longer stand for a virtual class, but rather (by analogy with a set function) a virtual-class *function*. It is transformed into an expression for a virtual class by replacing all occurrences of '*z*' by an individual constant (or a suitable Russellian description, if such is available).

In analogous fashion our notation may be extended to provide for virtual dyadic relations. Thus, we may define

$$\text{'}x\, zw \ni (\text{—}z\text{—}w\text{—})\, y\text{'} \text{ or } \text{'}zw \ni (\text{—}z\text{—}w\text{—})\, xy\text{'}$$

$$\text{'}x\, \{zw{:}\text{—}z\text{—}w\text{—}\}\, y\text{'} \quad \text{or} \quad \text{'}\{zw{:}\text{—}z\text{—}w\text{—}\}\, xy\text{'}$$

merely as

$$\text{'}(\text{—}x\text{—}y\text{—})\text{'},$$

where '(—*x*—*y*—)' is some formula of *L* containing '*x*' and '*y*' as its only free variables, '*z*' and '*w*' are any variables distinct from each other and from '*x*' and '*y*', and '(—*z*—*w*—)' differs from '(—*x*—*y*—)' appropriately. Similarly, we may go on to virtual triadic relations, and so on. The abstracts here are respectively of two and three places.

2. Why Bother? Why should we trouble ourselves about virtual classes and relations? First of all, we gain a "wealth of notation," as Quine puts it, "and we have seen how to define it in such a way as to recognize no such things as classes and relations at all except as a defined manner of speaking. A motive for talking thus ostensibly and eliminably of classes and relations is compactness of expression."[3] All the laws concerning virtual classes and relations turn out to be mere laws of the logic of quantification and identity in disguise, as we shall note in a moment. These laws are more compact, and often more intuitive, versions of the latter. Of course, "compactness of expression" is in general achieved by definitions in the real theory also. The more important motive here surely concerns ontic commitment. The various laws governing virtual entities in no way ontically commit us to such entities, whereas in the "real' formulation they do.

"Down the centuries," Quine notes, "a major motive, certainly, for assuming such objects as relations and classes or attributes has been this kind of [notational] convenience, and we now see that this kind of convenience can be served equally well by a virtual theory that assumes no such objects after all."[4] Also, there is little reason to think that down the centuries real classes and/or attributes were being aimed at any more than were virtual ones. For one thing, the very notion of being a value for a variable was not too clear. Of course one may have specific names for particular classes, relations, universals, patterns, forms, or whatever, without assuming that these names are in any way substitutable for variables. Thus one can often treat such objects virtually with impunity. Also, virtual classes of virtual classes can be introduced in various restricted ways—no doubt forms of forms, and so on. History should surely be viewed in the light of present knowledge. We have both the virtual and real theories now before us. A detailed argument from the text would no doubt be required to establish that any particular author (metaphysician, logician, or contemporary analyst) was aiming at the real, rather than the merely virtual, notion.

Another closely related point is that for philosophical purposes—especially in contemporary analytic philosophy—most of the reasons for having classes and relations at all seem served equally well by the virtual theory. Much philosophical writing is so inexact that the distinction between virtual and real seems premature. However, in more sophisticated writing, one speaks either of specific classes and relations or of classes and relations in general, where at most a large finite or a denumerable number is intended. Usually it is not clear

[3] *Set Theory and Its Logic*, p. 26.

[4] *Ibid.*, pp. 26–27.

which are to be regarded as primitive—expressions for them, that is—and which are not. No matter. Such entities, whether expressions for them are primitive or defined, need not be taken as values for variables, and this is the key point. After all, philosophers tend to be interested more in *specific* classes than in some general theory about all classes. Also, they tend to be more concerned with the *members* of classes than with the classes as such, and these latter in fact are usually thought of as being uniquely determined by their members. Whatever classes are, they are usually thought to be in some sense functions of, or at least dependent upon, their members. Further, philosophers often regard classes as the *extensions* of properties, in which case they are already in effect virtual. Similar remarks presumably hold when we turn to classes of classes.

In view of the consistency of first-order logic, no inconsistency can arise in the theory of virtual classes. Hence of course there is no possibility of constructing embarrassing classes that might lead to contradiction. Also, the virtual-class technique provides a very "natural" way of going about constructing classes, getting them out of their membership rather than, as it were, out of thin air.

We have spoken here of ontic commitment only relative to an object language. However, related comments apply to syntactical and semantical metalanguages as well. The convenience of virtual classes and relations can equally well be achieved in such metalanguages, where use is made of virtual classes of and relations between individuals as well as of or between expressions, and virtual relations between expressions and individuals. Here is an important matter that Quine seems not to have noted sufficiently. The wholesale use of the virtual-class technique makes possible the restricted semantical metalanguages based on denotation.

However, even if one does not wish to accept such metalanguages, there is still a connection between virtual classes and denotation. What kinds of terms can, in the most proper sense, be said to denote? According to the *Oxford English Dictionary*, *denotation* is "that which a word denotes . . . ; the aggregate of objects of which a word may be predicated." The use of 'aggregate' here is suggestive. An aggregate is a collection or agglomeration, or even a virtual class, but not of necessity a real one. The "words" involved here are words that "may be predicated" of objects. Now, to say that a term is predicable of an object is not to require that that term designate a real class; it is to require that that term be such as to stand significantly in the position of a predicate. Hence, that term can significantly be an expression for a virtual class. Such expressions are significantly predicable of objects,

just as expressions for real classes are, of course, but without the added commitment to existence, which is not involved in mere predication anyhow.

3. Some Useful Notions. Let us reflect, a little more fully than Quine does, upon the vast number of useful logical notions that may be construed virtually. Ordinarily it has been supposed that the definition of these notions presupposes the full resources of class or set theory. That this is not the case is occasion not only for some surprise but for rejoicing as well. For convenience in listing these ideas we shall use the terminology of *Principia Mathematica.*

The *inclusion* and *identity* of virtual classes are clearly definable in the usual way. A given virtual class is included in a virtual class if and only if every member of the one is a member of the other. Identical virtual classes are then those that mutually include one another. The various Boolean notions are readily definable. The *logical sum* of two virtual classes is the virtual class of all x's such that x is a member of one or the other, perhaps of both. Similarly the *logical product* of two virtual classes, the *negation* of a virtual class, the *universal* virtual class, and the *null* virtual class may be introduced. And similarly for relations. (See *20–*25 of *PM.*)

Russellian descriptions (*14), phrases of the form 'the one so-and-so', are of course definable without use of virtual classes or relations. *Descriptive functions* (*30), phrases of the form 'the one individual which bears R to y', are now definable where R is a given virtual dyadic relation. Given a virtual dyadic relation, its *converse* (*31) may be introduced. We may go on to *referents* and *relata* of a given term with respect to a given dyadic virtual relation (*32), to *domains, converse domains,* and *fields* of a virtual dyadic relation (*33), to the *relative product* of two dyadic virtual relations (*34), to dyadic virtual relations with *limited domains and converse domains* including *Cartesian products* (*35), to *plural descriptive functions* (*37), to the theory of *operations* (*38), to *unit classes and cardinal couples* (*51, *54), to *ordinal couples* (*55), to *one-many, many-one,* and *one-one* virtual dyadic relations (*71, *72, *74). The notions here are definable essentially as in *PM,* with some minor changes here or there, and most of the theorems given there are now provable for virtual classes and relations. The omitted notions and theorems are mainly of interest for the foundations of arithmetic and involve fundamentally higher logical types. Thus most of Volume I of *PM,* other than the material specifically concerned with mathematics, can readily be handled virtually.

We can press the theory of dyadic virtual relations still further, to include the notion of a *relation contained in diversity* (*200), of a *transitive*, *intransitive*, and *nontransitive relation* (*201), of a *symmetrical*, *asymmetrical*, and *nonsymmetrical* relation, of a *reflexive*, *irreflexive*, and *nonreflexive* relation, of a *connected* relation (*202), of a *serial* relation (*204), of a *partial ordering relation*, and of a *simple ordering relation*. No doubt there are other useful notions to be gained if we were to press further. This list is by no means complete, but let it suffice for the moment to convince us of the wealth of logical notions definable virtually.

If now we take into account the nonlogical primitive predicates of *L*, further notions are of course definable. As an example, suppose a relation of *discreteness* between individuals were available in *L* (either primitively or defined), in the sense that the individuals that are its arguments are spatio-temporal objects having no part in common.[5] Given a virtual class *F* of such entities, we say that an individual *x* is the *fusion* of *F* if and only if for every *z*, *z* is discrete from *x* if and only if *z* is discrete from every member of *F*. We say that the spatio-temporal individual *x* is a *part* of *y* if and only if every individual discrete from *y* is discrete from *x*. Then a spatio-temporal individual *x* is the *nucleus* of a virtual class *F* if and only if for every *z*, *z* is a part of *x* if and only if *z* is a part of every member of *F*. The theory of discreteness and attendant notions is contained in the so-called *calculus of individuals*. Strictly, the calculus concerns an uninterpreted relation of discreteness, whereas here we have been speaking of discreteness only as applied to spatio-temporal objects. The relation here is thus not strictly a relation of logic but presumably of the theory of space and time. In any event, the notions of fusion and nucleus are no doubt useful notions. And similarly for other virtually definable notions based on other nonlogical primitives.

4. Virtual Classes of Virtual Classes. Are there such things as virtual classes of virtual classes, or virtual classes of virtual relations, or virtual relations between virtual classes? Well, literally, no. However, a *manière de parler* for such can often be achieved in restricted contexts. We have noted above, for example, that the notion of a transitive virtual dyadic relation is definable. To say that *R* is transitive is merely to say that for any *x*, *y*, and *z*, if *xRy* and *yRz* then *xRz*. The very word 'transitive' here is in effect an expression for a virtual class of dyadic virtual relations. And similarly for many other virtual classes of or relations between or among virtual classes and/or relations.[6]

[5] See H. S. Leonard and N. Goodman, "The Calculus of Individuals and Its Uses," *The Journal of Symbolic Logic* 5 (1940), 45–55.

[6] Cf. again the methods used in "A Homogeneous System for Formal Logic."

We can explicitly introduce a notation for virtual classes of virtual classes by a straightforward adaptation of (1) or (2) of Section 1 above. Let '*F*' and '*G*' possibly with primes now be or abbreviate any one-place abstract. Pick out now some '*F*' containing no free variables and let it function as a second-order variable of abstraction, so to speak. Then

'*F* ∍ (—*F*—) *G*'

or

'{*F*:—*F*—} *G*'

may now be regarded as expressing that the virtual class *G* is a member of the virtual class of all classes *F* such that (—*F*—). And similarly for virtual dyadic relations between virtual classes, for which

'*G FF'* ∍ (—*F*—*F'*—) *G''* or '*FF'* ∍ (—*F*—*F'*—) *GG''*

or

'*G* {*FF'*:—*F*—*F'*—} *G''* or '{*FF'*:—*F*—*F'*—} *GG''*

give us a suitable notation. These abstracts are *second-order* abstracts. Similarly there are second-order abstracts for triadic relations, and so on. For all of these abstracts there are also appropriate principles of concretion and abstraction.

Further, second-order abstracts for virtual classes of virtual relations, for virtual relations between or among virtual relations or between or among virtual classes and virtual relations, may be introduced in similar fashion. Of course, the use of all such abstracts introduces nothing essentially new that is not already contained in first-order logic.

The theory of virtual classes and relations and first-order logic, we see then, are in effect one and the same. Arithmetic, and therewith mathematics and/or set theory are something in addition, to be provided for by specific nonlogical primitives. And similarly for any other deductive discipline. On this meaning for 'logic' the logistic thesis, that all mathematics is reducible to logic, is of course false, just as is the corresponding thesis concerning, say, quantum mechanics. A key problem for the philosophy of mathematics is then what nonlogical primitive or primitives to adopt, and with what entities as arguments, and what best to postulate concerning them. The set-theoretic approach to arithmetic is merely one out of many, but not necessarily the most satisfactory.

Note, incidentally, that the phrase 'there exists a virtual class such that. . .', although strictly meaningless, may still be used occasionally or in restricted contexts. To say, for example, that there exists a universal virtual class is to say in effect that there exists an abstract of such and such a kind denoting all objects. To say that such and such a virtual class exists is to speak in a semantical metalanguage to the effect that such and such an expression denotes such and such objects.

It might be thought that our theory is intimately linked with nominalism, but this is not, it would seem, the case. Nominalists avoid assuming that there are such things as real classes or relations. Hence of course the virtual technique is serviceable to the nominalist. That it is also of interest to the set-theoretic realist is shown in Quine's book, for example, where a realist position as to the existence of sets is taken hand in hand with an extensive use of virtual classes.

Also it might be thought that virtual classes are linked in some way with finitism. It might be presumed that finite sets are virtually definable by complete enumeration, whereas infinite sets are not. To presume this is to miss the crucial point that virtual classes are not values for variables, whereas real sets, finite or infinite, are.

5. Sets and Classes. Finally, we should consider a terminological recommendation: hereafter to use 'class' solely for virtual classes and 'set' for the real thing. When philosophers speak of classes it is usually, or at least often, a virtual class that is meant—or if not, what is said can usually be rephrased appropriately so as to involve only a virtual class. Mathematicians generally seem to prefer the word 'set' to 'class'. Sets are real mathematical entities, as are sets of sets, and so on. However, in the usage being recommended, classes are not, and the very phrase 'class of classes' is already a little barbarous, and in fact is meaningless according to some set theorists.

Quine is usually—and indeed unusually—fastidious in his terminology, but in *Set Theory and Its Logic* he has perhaps missed the opportunity to straighten out this terminology once and for all. Quine uses 'set' and 'class' "almost interchangeably," as he puts it, but follows von Neumann in holding that not all classes are capable of being members of classes. Those that are, are sets, those that are not, are "ultimate classes," as Quine calls them. The word 'set' has more currency than 'class' in mathematical contexts, but Quine favors the word 'class' to 'set' except in calling the whole subject 'set theory'. Why not use 'set' for sets, 'class' for virtual classes, and then 'ultimate set' for those special entities incapable of being members but that may have members? (These special entities may in fact be virtual classes, as in Bernays' system of 1958, as Quine points out, or they may be values for a special sort of variable, as in Bernays' earlier system.)[7] This terminology would be unambiguous and would help to remind philosophers and mathematicians of their proper concern with virtual classes and with sets or other mathematical objects, respectively. That there may be various kinds of sets, some of them "ultimate," need occasion no surprise. Nonetheless, the notion of an ultimate set is surely *ad hoc*, introduced merely to help avoid contradictions, and its artificiality adds fuel to the arguments of those who are suspicious of the mathematical notion of set anyhow. A satisfactory account of the notion of a virtual class, however, may be given quite independently, as we have seen, an account that seems adequate for purposes other than those of founding mathematics. For this latter task we should perhaps look to notions other than that of set, so that even in mathematics the virtual technique may perhaps be made to suffice. This, however, is quite another topic.

Let us turn again to the topic of belief, with virtual classes now at hand.

6. Virtual Classes and Belief. It should be noted that in all the analyses of 'B' and 'K' given in the preceding chapters, B and K are regarded as $(n + k + 4)$-adic relations, depending on the number of arguments. One of these arguments, for the condition of belief, is a linguistic expression, and that this be the case is essential for all of the analyses thus far. It seems very difficult to see how the experimenter E can test X's beliefs completely other than by taking into account X's behavior as well as his reactions to certain sentences of some language L. Surely if beliefs are to be put to the test, as has been emphasized throughout, some such dependence on X's reactions to language seems essential. X may or may not know L, in which case his reactions are studied by E in some language known to X.

We have noted above that Church, who seems to advocate analyzing belief in terms of "propositions," wants beliefs to be empirically testable. However, X's belief concerning some proposition can be tested, it would seem, only by reference to some sentence that that proposition "expresses." In short, all analyses of belief seem to involve relativity in some fashion or other to the language in which the belief is formulated; the condition of belief must be expressed in some language or other as object language.

Let us consider now another handling of belief in which no

[7] P. Bernays, "A System of Axiomatic Set Theory," *The Journal of Symbolic Logic* 2 (1937), 65–77; 6 (1941), 1–17; 7 (1942), 65–89, 133–145; 8 (1943), 89–106; 13 (1948), 65–79; and 19 (1954), 81–96; and P. Bernays and A. A. Fraenkel, *Axiomatic Set Theory* (*Studies in Logic and the Foundations of Mathematics*, Amsterdam: North-Holland Publishing Co., 1958).

relativization to language is explicitly indicated in the definiendum. In other words, a definition of 'B' will be proposed *with no sentence or sentential function as an argument* for the condition of belief, but rather a suitable virtual class. This does not mean that relativization to language is avoided in the sense that the relation Acpt with a sentence as one of its arguments plays no role in the definiens. To the contrary, here as elsewhere the experimenter tests *X*'s belief by taking into account *X*'s acceptance or rejection of some sentence.

Consider again the case of a sentence expressing *X*'s belief that contains only one primitive predicate constant and only one primitive individual constant. Let '(—*x*—)' as above be some sentential function of *L* containing just one variable '*x*'. We then let

$$`X \ \mathrm{B}_E \ \boldsymbol{P}, y, x \ni (—x—), t'$$

be short for

> 'There are expressions *b*, *c*, and *d* of *L* such that (1) for all *z*, *b* Den *z* if and only if ***P****z*, (2) *c* PrNm *y*, (3) *b* is a primitive one-place predicate constant of *L* and the only such occurring in ***a***, (4) *c* is the only primitive individual constant of *L* occurring in ***a***, (5) *d* differs from ***a*** only in containing occurrences of the individual constant *c* wherever there are free occurrences of the variable '*x*' in ***a***, and (6) *X* Acpt_E *d*,*t*',

where in place of '***a***' we put in the structural-descriptive name of '(—*x*—)' and in place of '***P***' any primitive one-place predicate constant.

This again is not a definition, but rather a definition schema covering as it were an infinity of definitions. For specific ***P*** and '(—*x*—)', say 'P' and '(P*x* V ∼P*x*)', the definiens here may be read '*X* believes at *t* (according to *E*) of P's and of *y* that *y* is a member of the virtual class *x*∍(P*x* V ∼P*x*)'.

This definition schema may readily be generalized to sentences containing any finite number of primitive individual or predicate constants, as in Chapter V, Section 11.

The generalization of this definition schema as well as that of Chapter V, Section 11, are thought to be of especial interest. In fact, all the preceding analyses of 'B' may now be regarded merely as heuristics for these two. Of the two, the present one has the greater interest. It embodies completely the idea that only nonlinguistic entities be referred to in the definienda, even if only nominally so.

Suppose for the moment that *X* does not know *L* at all. Let us

be a little more explicit about this case than above. Suppose also that there is some *L′* that he does know, and which of course is known also to *E*. *E* then can test *X*'s reactions to sentences of *L′*. To state these results, *E* here as above employs a metalanguage of *L′*. *E* may use a metalanguage constructed on the basis of *L′* so that it contains *L′* as a part, or it may be constructed so as to contain translations of the expressions of *L′*. In the former case, the definition schema just given would be altered by referring to *L′* in place of *L*. In the latter, we not only refer to *L′* in place of *L*, but also change the proviso as follows: '(—*x*—)' is now regarded as a sentential function of the translational part of the metalanguage (not of *L′* as before), and in place of '***a***' we put in the structural-descriptive name of the translation of '(—*x*—)' into *L′*. In this way, then, *E* may express the results of his investigations concerning *X*'s beliefs in a metalanguage containing a translation of a language that *X* is presumed to know.

The situation here is in contrast to that of Chapter V, Section 4 and 10, above, where the sentences constituting the conditions of belief were sentences within the object language presumed known to *X*. Here the '(—*x*—)' is a sentential function within the metalanguage within which the object language occurs as a part or is suitably translated.

The notion of translation employed here is esentially that of Tarski, in *Der Wahrheitsbegriff*. Even if one were to obejct to its use, this would not impair the foregoing analysis. It would merely limit *E* to employing a metalanguage constructed in such a way as to contain as a part a language known to *X*.

A similar definition schema may be given for 'K' in terms of the semantical truth concept. Of course these schemata may be generalized so that ***a*** may contain any fixed finite number of primitive constants.

Later, in Chapter IX, Section 9, the definition of 'B' just given will be further improved and generalized. Meanwhile it may be regarded as the most satisfactory yet, supplanting those of preceding chapters. We shall be able to do better later, however, after discussing intensions, facts, propositions, events, and related topics.